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Interpolation of several multiparametric approximation spaces

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Abstract

We prove a general interpolation theorem for linear operators acting simultaneously in several approximation spaces which are defined by multiparametric approximation families. As a consequence, we obtain interpolation results for finite families of Besov spaces of various types including those determined by a given set of mixed differences.

1. Introduction

The main objective of the present paper is a general interpolation theorem for finite sets of approximation spaces determined by a multiparametric approximation families. Spaces of this kind are found in several areas of analysis including Sobolev type embeddings, linear and nonlinear multivariate approximation, and interpolation space theory. Typical examples are "mixed" $L_{\bar{p}}(\mathbb{R}^d)$ spaces with $\bar{p} := (p_1, \dots, p_d)$, anisotropic Sobolev or Besov spaces and spaces defined by a "dominated" set of mixed derivatives or differences. Some of the results related to these spaces are simple consequences of the respective one-dimensional facts and induction on dimension. However, genuine multidimensional results have nothing in

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common with this approach, since most of them are not true for univariate functions. Typical ones are inequalities for Fourier transform of a function from anisotropic Sobolev space $W_1^{\bar{r}}(\mathbb{R}^d)$ due to Bourgain [Bo], Pelczynski and collaborators [PSe,PW], and Kolyada (see survey [K, Section 13.3]).

Unfortunately, proofs of results of this type are mostly long and complicated; we hope that interpolation space technique presented in this paper may be of importance for clarifying these proofs and obtaining new results in this area.

The basic concept, an approximation space, appeared as a by-product of the classical S. Bernstein–Jackson theory of trigonometrical approximation (see, e.g., [N,T] and references therein). Abstract approximation spaces were introduced by Brudnyi and Timan (1959) for the special case of monotone families of linear subsets $\{A_k\}$ with $\dim A_k = k,\ k \in \mathbb{Z}_+$ (see references in [T]). More general one-parametric approximation spaces were then introduced and studied by Peetre and Sparr [PS] (see [BK2, Section 4.2] for references of subsequent papers by I. Asekritova, Yu. Brudnyi, N. Krugljak, P. Nilsson and A. Pietsch and others devoted to the topic).

The approximation space of the present paper, $E_\Phi(X;\mathcal{A})$, is determined by the next three ingredients: an *ambient Banach space* X, *an approximation family* $\mathcal{A} := \{A_k \subset X : k \in \mathbb{Z}_+^d\}$ and an *E-parameter* Φ, which is a Banach lattice of functions on $\mathbb{Z}_+^d$ (see Section 2 for these and consequent notions and notations).

The main problem of the paper is as follows:

Let $E_{\bar{\Phi}}(\bar{X};\mathcal{A}) := (E_{\Phi_i}(X_i;\mathcal{A}))_{i=0}^n$ be an *n-tuple of approximation spaces* and $\mathcal{F}$ be an *interpolation functor on a category of Banach n-tuples*.

Problem. *Find conditions on $\mathcal{A}$ and $\mathcal{F}$ ensuring validity of the equality*

$$\mathcal{F}(E_{\bar{\Phi}}(\bar{X};\mathcal{A})) = E_{\mathcal{F}(\bar{\Phi})}(\mathcal{F}(\bar{X});\mathcal{A}). \tag{1.1}$$

It is not difficult to derive from (1.1) the *splitting property* of $\mathcal{F}$ with respect to the n-tuple $\bar{\Phi}(\bar{X}) := (\Phi_i(X_i))_{i=0}^n$ of the vector-valued Banach lattices. We will show that this property is also sufficient for (1.1) if, in addition, $\mathcal{A}$ is *complemented* in $\bar{X}$ and has a special algebraic structure. This allows us to derive new interpolation results for n-tuples of anisotropic Besov spaces and similar objects of this kind. Some of them are new even for the well-studied case of couples ($n=1$). For example, an anisotropic analog of the classical real interpolation theorem asserts that

$$(B_{p_0}^{\bar{s}^0,q_0}(\mathbb{R}^d),\ B_{p_1}^{\bar{s}^1,q_1}(\mathbb{R}^d))_{\theta q} = B_p^{\bar{s}}(\mathbb{R}^d) \tag{1.2}$$

for $\bar{s}, p, q$ being, respectively, θ-*means* of $(\bar{s}^0,\bar{s}^1)$, (p_0,p_1) and (q_0,q_1), provided that $q=p$ and $\bar{s}^0,\bar{s}^1$ are collinear, see [G,Tr, Section 2.13]. The latter condition looks unnatural; as in many other cases in this area it, in fact, follows from a presentation of $B_p^{\bar{s},q}$ as a *one-parametric* approximation space. A d-parametric representation allows us to find the real and complex interpolation spaces for n-tuples of arbitrary anisotropic Besov spaces as d-parametric approximation spaces and investigate the cases of validity of a relation similar to (1.2). A partial case is pointed out in Section 4.2; the general situation will be presented elsewhere.

The proof of the main result is based on the *decomposition method* going back to classical Bernstein's proof of his inverse theorem. For the first time it was used in [BK1], where, in particular, a new proof of the Peetre–Sparr interpolation theorem [PS] was given. The method and some results of this rather unaccessible paper are presented in [BK2, Section 4.2, P]. Complemented approximation families were introduced, under the name of VP-systems, by the second named author in order to prove (1.1) for Banach couples of approximation spaces, see [BSh] and also [BK2, Theorem 4.3.2]. The decomposition method for nonlinear complemented families were then studied in [DP].

The interpolation theorem (1.1) has many interesting applications in analysis, a detailed account of which is out of the content of the present paper. We intend to discuss this subject elsewhere.

2. Preliminaries

We will introduce and briefly discuss the basic concepts involved in our consideration.

2.1. Approximation families

Let $\mathcal{A} := \{A_k : k \in \mathbb{Z}_+^d\}$ be a collection of *linear subsets* in a Banach space X indexed by d-vectors of nonnegative integer coordinates. The index set is assumed to be ordered by the coordinate-wise order, that is to say, $k \leqslant \ell$, if $k_i \leqslant \ell_i$ for $1 \leqslant i \leqslant d$, and $k < \ell$, if, in addition, $k \neq \ell$.

Definition 2.1. $(X, \mathcal{A})$ is said to be a *d-parametric approximation family* (briefly, *AF*), if for $k < \ell$

$$\{0\} = A_0 \subset A_k \subsetneqq A_\ell \, . \tag{2.1}$$

Given the $(X, \mathcal{A})$, the *approximation number of the order* k is introduced by

$$e_k(x; X) := \inf\{||x - a||_X : a \in A_k\}. \tag{2.2}$$

These, in turn, define a sublinear operator $e_{\mathcal{A}} : X \to \ell_\infty(\mathbb{Z}_+^d)$ given by

$$e_{\mathcal{A}}(x; X) := (e_k(x; X))_{k \in \mathbb{Z}_+^d}. \tag{2.3}$$

It is worth noting that

$$e_{\mathcal{A}}(X) \subset m(\mathbb{Z}_+^d), \tag{2.4}$$

where $m(\mathbb{Z}_+^d)$ is the cone of bounded nonnegative nonincreasing functions on $\mathbb{Z}_+^d$.

Definition 2.2. $(X, \mathcal{A})$ is said to be *complemented*, if there exists a family $\mathcal{P}$ of linear projectors $P_k : X \to A_k$, $k \in \mathbb{Z}_+^d$ such that their norms are uniformly bounded.

In the sequel we use the notation

$$\|\mathcal{P}\|_X := \sup_k \|P_k\|_X. \tag{2.5}$$

In applications a variant of this notation will be useful.

Definition 2.3. $(X, \mathcal{A})$ is said to be *quasicomplemented*, if there exists a family $\mathcal{P}$ of linear operators $P_k : X \to A_{k+1},\ k \in \mathbb{Z}_+^d$ satisfying (2.5) and such that

$$P_k x = x, \quad \text{if} \quad x \in A_k. \tag{2.6}$$

Hereafter $k+1$ stands for $(k_1+1, \dots, k_d+1)$.

Approximation numbers relate to the projectors (or quasiprojectors) P_k by the following (Lebesgue) inequality:

$$\|x - P_k x\|_X \leqslant (1 + \|\mathcal{P}\|_X) e_k(x; X). \tag{2.7}$$

On the other hand, the left-hand side is bounded below by $e_{k+1}(x; X)$ or $e_k(x; X)$, if P_k is, respectively, a quasiprojector and projector.

2.2. E-parameters

The next ingredient of the basic concept, an *approximation space*, is introduced by

Definition 2.4. *E-parameter* Φ is a Banach lattice (a.k.a. ideal space) of functions $f : \mathbb{Z}_+^d \to \mathbb{R}$ such that

$$\ell_0(\mathbb{Z}_+^d) \subset \Phi \subset \ell_\infty(\mathbb{Z}_+^d) \tag{2.8}$$

and, in addition, for each bounded sequence $\{f_j\} \subset \Phi$

$$\| \varliminf_{j\to\infty} f_j \|_\Phi \leqslant \varliminf_{j\to\infty} \|f_j\|_\Phi. \tag{2.9}$$

Here $\ell_0(\mathbb{Z}_+^d)$ is the space comprising bounded functions of *finite support*. By the closed graph theorem the right embedding in (2.8) is, in fact, *continuous*. The property (2.9) means that the closed ball of Φ is closed under pointwise convergence; this is usually called the *Fatou property*.

2.3. Approximation spaces (AS)

Given an approximation family $(X, \mathcal{A})$ and the E-parameter Φ associated with the index set $\mathbb{Z}_+^d$, we now introduce the basic concept.

Definition 2.5. *Approximation space* $E_\Phi(X;\mathcal{A})$ is a linear subset of X determined through finiteness of the norm

$$||x||_{E_\Phi(X;\mathcal{A})} := ||e_{\mathcal{A}}(x;X)||_\Phi. \tag{2.10}$$

A straightforward consequence of sublinearity of $e_{\mathcal{A}}$ and the Fatou property of Φ is the following statement:

Proposition 2.6. *$E_\Phi(X;\mathcal{A})$ is a Banach space continuously embedded in X.*

2.4. Category of Banach n-tuples $\mathcal{B}_n$

In order to formulate and prove our main result, we need several notions of Interpolation Space Theory, see, e.g., [BK2, Chapter 2] for a detailed account.

A *Banach n-tuple* $\bar{X} := (X_0, X_1, \dots, X_n)$ is an ordered sequence of Banach spaces continuously embedded in a Hausdorff topological linear space. The *sum* and *intersection* of $\bar{X}$ is defined by

$$\Sigma(\bar{X}) := X_0 + \cdots + X_n, \quad \Delta(\bar{X}) := \bigcap_{i=0}^{n} X_i.$$

They are Banach spaces under the norms

$$||x||_{\Sigma(\bar{X})} := \inf\left\{\sum_{i=0}^{n} ||x_i||_{X_i} : x = \sum_{i=o}^{n} x_i\right\},$$
$$||x||_{\Delta(\bar{X})} := \max\{||x||_{X_i} : 0 \leqslant i \leqslant n\}.$$

A Banach space X is said to be *intermediate* for $\bar{X}$, if

$$\Delta(\bar{X}) \hookrightarrow X \hookrightarrow \Sigma(\bar{X}).$$

Hereafter $X \hookrightarrow Y$ means that a Banach space X is algebraically and topologically embedded in a Banach space Y.

A linear map $T : \Sigma(\bar{X}) \to \Sigma(\bar{Y})$ is called a *linear continuous operator* from $\bar{X}$ into $\bar{Y}$, if for each $0 \leqslant i \leqslant n$

$$T(X_i) \subset Y_i.$$

By the closed graph theorem the norm

$$||T||_{\bar{X},\bar{Y}} := \max\{||T|_{X_i}||_{X_i,Y_i} : 0 \leqslant i \leqslant n\} \tag{2.11}$$

is finite.

The linear space of all these T equipped with norm (2.11) is Banach; it is denoted by $L(\bar{X}, \bar{Y})$ or simply $L(\bar{X})$, if $\bar{Y} = \bar{X}$.

The collection of n-tuples forms the *class of objects* for category $\mathcal{B}_n$, while the collection of linear continuous operators acting between n-tuples constitutes the *class of its morphisms*. Particularly, $\mathcal{B}_0$ is a category of Banach spaces and linear continuous operators acting between them, while $\mathcal{B}_1$ is a category of *Banach couples*, the most developed object of Interpolation Space Theory.

An *interpolation functor* $\mathcal{F}$ on $\mathcal{B}_n$ maps this category into $\mathcal{B}_0$ such that $\mathcal{F}(\bar{X})$ is an intermediate space of $\bar{X}$ and $\mathcal{F}(T)$ for $T \in L(\bar{X}, \bar{Y})$ is the restriction of T to $\mathcal{F}(\bar{X})$. This definition implies the interpolation inequality

$$||T|_{\mathcal{F}(\bar{X})}||_{\mathcal{F}(\bar{X}),\mathcal{F}(\bar{Y})} \leqslant C||T||_{\bar{X},\bar{Y}} \tag{2.12}$$

with $C \geqslant 1$ independent of $\bar{X}$, $\bar{Y}$ and T. The *optimal* C is denoted by $C_{\mathcal{F}}$.

The functor is *exact* if $C_{\mathcal{F}} = 1$. Trivial examples of exact interpolation functors are $\Sigma : \bar{X} \to \Sigma(\bar{X})$, $\Delta : \bar{X} \to \Delta(\bar{X})$ and $\pi_i : \bar{X} \to X_i$, $0 \leqslant i \leqslant n$. More substantive examples will be introduced below.

At last, we consider a simple property of interpolation functors that will be used in the sequel. To this end let us define *the direct sum* of Banach spaces X, Y as the linear space

$$X \oplus Y := \{(x, y) : x \in X,\ \ y \in Y\}$$

equipped with the (Banach) norm

$$||(x, y)||_{X \oplus Y} := ||x||_X + ||y||_Y.$$

Using this we define the direct sum of two n-tuples by

$$\bar{X} \oplus \bar{Y} := (X_0 \oplus Y_0, \ldots, X_n \oplus Y_n).$$

Let now $\mathcal{F}$ be an interpolation functor on $\mathcal{B}_n$.

Proposition 2.7. *Up to equivalence of the norms*

$$\mathcal{F}(\bar{X} \oplus \bar{Y}) = \mathcal{F}(\bar{X}) \oplus \mathcal{F}(\bar{Y}) \tag{2.13}$$

with the constant of equivalence depending only on $C_{\mathcal{F}}$. *In particular*, (2.13) *is an isometry*, *if* $\mathcal{F}$ *is exact*.

Proof. It is easily checked that

$$\Sigma(\bar{X} \oplus \bar{Y}) = \Sigma(\bar{X}) \oplus \Sigma(\bar{Y}). \tag{2.14}$$

Then the canonical injection $i_{\bar{X}} : \bar{X} \to \bar{X} \oplus \bar{Y}$ maps $\Sigma(\bar{X})$ into $\Sigma(\bar{X} \oplus \bar{Y}) = \Sigma(\bar{X}) \oplus \Sigma(\bar{Y})$ and therefore $\mathcal{F}(i_{\bar{X}}) = i_{\bar{X}}|_{\mathcal{F}(\bar{X})}$ is an *injection* of $\mathcal{F}(\bar{X})$ into $\mathcal{F}(\bar{X} \oplus \bar{Y})$. Similarly, the canonical projection $p_{\bar{X}} : \bar{X} \oplus \bar{Y} \to \bar{X}$ maps $\Sigma(\bar{X} \oplus \bar{Y}) = \Sigma(\bar{X}) \oplus \Sigma(\bar{X})$ onto $\Sigma(\bar{X})$, and therefore $\mathcal{F}(p_{\bar{X}}) = p_{\bar{X}}|_{\mathcal{F}(\bar{X} \oplus \bar{Y})}$ is a projection of $\mathcal{F}(\bar{X} \oplus \bar{Y})$ onto $\mathcal{F}(\bar{X})$. Besides, $p_{\bar{X}} \circ i_{\bar{X}} = 1_{\bar{X}}$ and therefore $\mathcal{F}(p_{\bar{X}}) \circ \mathcal{F}(i_{\bar{X}}) = 1_{\mathcal{F}(\bar{X})}$. The same is true for the canonical mappings $p_{\bar{Y}}$ and $i_{\bar{Y}}$. Hence $\mathcal{F}(i_{\bar{X}})$ and $\mathcal{F}(i_{\bar{Y}})$ are the injections of, respectively, $\mathcal{F}(\bar{X})$ and $\mathcal{F}(\bar{Y})$ in $\mathcal{F}(\bar{X} \oplus \bar{Y})$, while $\mathcal{F}(p_{\bar{X}})$ and $\mathcal{F}(p_{\bar{Y}})$ are the projections of $\mathcal{F}(\bar{X} \oplus \bar{Y})$ on, respectively, $\mathcal{F}(\bar{X})$ and $\mathcal{F}(\bar{Y})$. Besides, the corresponding products are the identities of $\mathcal{F}(\bar{X})$ and $\mathcal{F}(\bar{Y})$. This immediately implies (2.13). $\square$

2.5. Splitting interpolation functors

Let $\mathcal{E}_n(\mathbb{Z}_+^d)$ be a subcategory of $\mathcal{B}_n$ comprising n-tuples of E-parameters as its objects. Given $\bar{\Phi}\in\mathcal{E}_n(\mathbb{Z}_+^d)$ and $\bar{X}\in\mathcal{B}_n$, one introduces the n-tuple

$$\bar{\Phi}(\bar{X}) := (\Phi_0(X_0), \ldots, \Phi_n(X_n)). \tag{2.15}$$

Here $\Phi(X)$ is a Banach space of vector-valued functions $f : \mathbb{Z}_+^d \to X$ given by

$$||f||_{\Phi(X)} := ||(||f(k)||_X)_{k\in\mathbb{Z}_+^d}||_\Phi \tag{2.16}$$

Definition 2.8. Interpolation functors $\mathcal{F}$ on $\mathcal{B}_n$ *splits* $\bar{\Phi}(\bar{X})$ if

$$\mathcal{F}(\bar{\Phi}(\bar{X})) = \mathcal{F}(\bar{\Phi})(\mathcal{F}(\bar{X})) \tag{2.17}$$

with equivalence of the norms.

Several important functors on $\mathcal{B}_1$ possessing this property were discovered by A. Calderòn and Lions and Peetre (see, e.g., [BK2, Section 4.3] and references therein). The Calderòn theorem asserts the splitting property for the *upper complex functor* $\mathcal{C}^\eta$, $0<\eta<1$, and gives a constructive description of the space $\mathcal{C}^\eta(\bar{\Phi})$, $\bar{\Phi} := (\Phi_0, \Phi_1)$. In the case of $\bar{\Phi}\in\mathcal{E}_1(\mathbb{Z}_+^d)$ it implies that

$$\mathcal{C}^\eta(\bar{\Phi}(\bar{X})) = \Phi(\mathcal{C}^\eta(\bar{X})), \tag{2.18}$$

where

$$\Phi := \Phi_0^{1-\eta}\,\Phi_1^\eta \tag{2.19}$$

is the *Calderòn operation* on couples $\bar{\Phi}$ of Banach lattices. Let us recall that Φ is a Banach lattice defined through the norm

$$||f||_\Phi := \inf\{||f_0||_{\Phi_0}^{1-\eta}\,||f_1||_{\Phi_1}^\eta : |f| = |f_0|^{1-\eta}\,|f_1|^\eta\}. \tag{2.20}$$

It is worth noting that Φ is also an E-parameter. In fact, condition (2.8) is clearly true for Φ. The Fatou property of Φ is known to be equivalent to the duality relation $\Phi'' = \Phi$, where Φ' is the Banach lattice *associated* with Φ. In our case Φ' is defined through the norm

$$||f||_{\Phi'} := \sup\left\{\left|\sum_{k\in\mathbb{Z}_+^d} f(k)g(k)\right| : ||g||_\Phi \leqslant 1\right\}.$$

According to the general duality theorem, due to Lozanovski [L],

$$(\Phi_0^{1-\eta}\,\Phi_1^\eta)' = (\Phi_0^1)^{1-\eta}(\Phi_1')^\eta.$$

Hence Φ has the required property, provided Φ_0, Φ_1 have.

We now introduce the $\mathcal{C}^\theta$ on $\mathcal{B}_n$, $\theta := (\theta_0, \ldots, \theta_n)$, where $\theta_i > 0$ and $\sum_{i=0}^n \theta_i = 1$, using the following general iterative procedure.

Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be interpolation functors on the category of Banach couples $\mathcal{B}_1$. One introduces an interpolation functor $\mathcal{F} := \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n$ on $\mathcal{B}_n$ inductively by setting

$$(\mathcal{F}_1 \times \cdots \times \mathcal{F}_n)(\bar{X}) := \mathcal{F}_1(X_0, (\mathcal{F}_2 \times \cdots \times \mathcal{F}_n)(X_1, \ldots, X_n)). \tag{2.21}$$

It is easily verified that (2.20) defines an interpolation functor on $\mathcal{B}_n$ with the interpolation constant bounded by $\prod_{i=1}^{n} C_{\mathcal{F}_i}$. To clarify the choice of parameters in our definition of $\mathcal{C}^\theta$ as a product (2.19), we introduce the following notion.

Interpolation functor $\mathcal{F}$ on $\mathcal{B}_n$ is of the *power type* $\theta := (\theta_0, \ldots, \theta_n), \theta_i > 0,\ \sum_{i=0}^{n} \theta_i = 1$ if for each $T \in L(\bar{X}, \bar{Y})$ the following interpolation inequality:

$$\|T|_{\mathcal{F}(\bar{X})}\|_{\mathcal{F}(\bar{X}),\mathcal{F}(\bar{Y})} \leqslant C \prod_{i=0}^{n} (\|T|_{X_i}\|_{X_i,Y_i})^{\theta_i} \tag{2.22}$$

holds with a constant independent of T.

Proposition 2.9. *If $\mathcal{F}_i$ is an interpolation functor on $\mathcal{B}_1$ of the power type $(1 - \eta_i, \eta_i)$, $1 \leqslant i \leqslant n$, then the functor in (2.21) is of the power type $\theta = (\theta_0, \ldots, \theta_n)$ where*

$$\theta_i := \eta_0\eta_1 \cdots \eta_i(1 - \eta_{i+1}); \tag{2.23}$$

here $\eta_0 := 1$ and $\eta_{n+1} := 0$.

Proof. Use induction on n. $\square$

This proposition motivates our next

Definition 2.10. Let $\theta := (\theta_0, \ldots, \theta_n)$, $\theta_i > 0, \sum_{i=0}^{n} \theta_i = 1$, and η_i, $1 \leqslant i \leqslant n$ be the (unique) solution to the system of equations (2.23). Then the *upper complex functor* $\mathcal{C}^\theta$ is given by

$$\mathcal{C}^\theta := \mathcal{C}^{\eta_1} \times \mathcal{C}^{\eta_2} \times \cdots \times \mathcal{C}^{\eta_n}. \tag{2.24}$$

For the case of n-tuples of E-parameters $\mathcal{C}^\theta(\bar{\Phi})$ can be computed through the Calderón operation $\bar{\Phi}^\theta := \Phi_0^{\theta_0} \cdots \Phi_n^{\theta_n}$ which is introduced similarly to (2.20). Actually, according to the Calderón theorem, $\mathcal{C}^\eta(\Phi_0, \Phi_1) = \Phi_0^{1-\eta}\Phi_1^\eta$ with the constant of equivalence for the norms bounded by 2. Therefore induction on n straightforwardly yields the relation

$$\mathcal{C}^\theta(\bar{\Phi}) = \bar{\Phi}^\theta \tag{2.25}$$

with the constant of equivalence bounded by 2^n. In turn, the splitting result (2.18) combined with induction on n immediately gives

Proposition 2.11. *It is true that*

$$\mathcal{C}^\theta(\bar{\Phi}(\bar{X})) = \bar{\Phi}^\theta(\mathcal{C}^\theta(\bar{X})) \tag{2.26}$$

provided $\bar{\Phi} \in \mathcal{E}_n(\mathbb{Z}_+^d)$ and $\bar{X} \in \mathcal{B}_n$.

Applying construction (2.21) to the real interpolation functors $\bar{X}_{\eta_i,q}$ with $\eta_i \in (0,1)$ given by (2.23), one then can use the obtained functor to set the respective splitting result. The starting point here is the Lions–Peetre theorem which, in particular, implies that

$$(\bar{\Phi}(\bar{X}))_{\eta,q} = \Phi(\bar{X}_{\eta,q});$$

here $\Phi := \Phi_0^{1-\eta}\Phi_1^{\eta}$ and E-parameters Φ_0, Φ_1 are weighted ℓ_p spaces with *distinct* p's. Unfortunately, the last condition brings unnecessary restrictions into the final result. For this reason we prefer to use another generalization of the real functor introduced through the notion of *K-functional* on $\mathcal{B}_n$. Let us recall its definition: for $x \in \Sigma(\bar{X})$ and $t \in \mathbb{R}_+^{n+1}$

$$K(t; x; \bar{X}) := \inf\left\{\sum_{i=0}^{n} t_i \|x_i\|_{X_i} : x = \sum_{i=0}^{n} x_i\right\}. \tag{2.27}$$

The required functor on $\mathcal{B}_n$ is denoted by $\mathcal{R}_{\theta q}$, $\theta_i > 0$, $\sum_{i=0}^{n} \theta_i = 1$, $1 \leqslant q \leqslant \infty$, and introduced through the norm

$$\|x\|_{\mathcal{R}_{\theta q}(\bar{X})} := \left\{\int_{\mathbb{R}_+^{n+1}} \left(\frac{K(t; x; \bar{X})}{t^{\theta}}\right)^q dH\right\}^{\frac{1}{q}}, \tag{2.28}$$

where $t^{\theta} := \prod_{i=0}^{n} t_i^{\theta_i}$, and $dH := \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n}$ is the (Haar) measure on $(\mathbb{R}_+ \backslash \{0\})^{n+1}$.

The splitting result for $\mathcal{R}_{\theta q}$ was due to Sparr [Sp, Theorem 8.2]. In the case of the E-parameters Φ_i given by norms of the form

$$\|f\|_{\ell_p^w} := \left\{\sum_{k \in \mathbb{Z}_+^d} (w(k)|f(k)|)^p\right\}^{\frac{1}{p}} \tag{2.29}$$

with $1 \leqslant p \leqslant \infty$ and a positive weight w, the Sparr theorem gives the equality

$$\mathcal{R}_{\theta q}(\bar{\Phi}(\bar{X})) = \bar{\Phi}^{\theta}(\mathcal{R}_{\theta q}(\bar{X})), \tag{2.30}$$

provided $\Phi_i := \ell_{p_i}^{w_i}$. It is worth noting that in this case

$$\bar{\Phi}^{\theta} = \ell_q^w, \tag{2.31}$$

where $w := \prod_{i=0}^{n} w_i^{\theta_i}$ and $\frac{1}{q} := \sum_{i=0}^{n} \frac{\theta_i}{p_i}$.

2.6. n-tuples of approximation spaces

Let $\bar{X} \in \mathcal{B}_n$ and $\mathcal{A} := \{A_k : k \in \mathbb{Z}_+^d\}$ be an approximation family in $\Sigma(\bar{X})$. *We say that $\mathcal{A}$ is AF in $\bar{X}$, if each $X_i \cap \mathcal{A}$ is AF in X_i,* $0 \leqslant i \leqslant n$. Here

$$X \cap \mathcal{A} := \{X \cap A_k : k \in \mathbb{Z}_+^d\}.$$

If now $\bar{\Phi}\in\mathcal{E}_n(\mathbb{Z}_+^d)$, then we introduce an n-tuple of approximation spaces by

$$E_{\bar{\Phi}}(\bar{X};\mathcal{A}) := (E_{\Phi_0}(X_0;X_0\cap\mathcal{A}),\dots,E_{\Phi_n}(X_n;X_n\cap\mathcal{A})). \tag{2.32}$$

Similarly to Definitions 2.2 and 2.3 we introduce

Definition 2.12. $(\bar{X},\mathcal{A})$ is *complemented* if there is a family $\mathcal{P} := \{P_k\in L(\bar{X}) : k\in\mathbb{Z}_+^d\}$ such that $P_k|_{X_i}$ is a projection on $X_i\cap A_k,\ 0\leqslant i\leqslant n,\ k\in\mathbb{Z}_+^d$, and, besides,

$$||\mathcal{P}||_{\bar{X}} := \sup_k ||P_k||_{\bar{X}} < +\infty. \tag{2.33}$$

Replacing here $P_k|_{X_i}$ by a mapping on $A_{k+1}\cap X_i$ preserving A_k, we introduce the notion of *quasicomplemented AF* $(\bar{X},\mathcal{A})$.

3. The main result

The result concerns approximation spaces generated by AF's of a special form. In order to introduce them we use a set $\bar{\mathcal{A}} := \{\mathcal{A}^i : 1\leqslant i\leqslant d\}$ of AF's in an n-tuple $\bar{X}$ and a *covering* κ of $\mathbb{N}(d) := \{1,\dots,d\}$ by (ordered) subsets which do not contain each other. To simplify the notations we assume all the $\mathcal{A}^i$ to be one-parametric, i.e., $\mathcal{A}^i := \{A_k^i : k\in\mathbb{Z}_+\}$; the general case can be derived by the very same argument. Given $\bar{\mathcal{A}}$ and κ, one introduces the desired $\mathcal{A} := \{A_k : k\in\mathbb{Z}_+^d\}$ by

$$A_k := \bigcap_{\omega\in\kappa}\left(\sum_{i\in\omega} A_{k_i}^i\right), \quad k=(k_1,\dots,k_d)\in\mathbb{Z}_+^d. \tag{3.1}$$

Let now $\bar{\Phi}\in\mathcal{E}_n(\mathbb{Z}_+^d)$ and $\omega\subset\mathbb{N}(d)$. Then one defines a subtuple $\bar{\Phi}_\omega$ of $\bar{\Phi}$ as follows. For the E-parameter Φ associated with $\mathbb{Z}_+^d$ one denotes by Φ_ω its (closed) subspace comprising functions of variables k_i with $i\in\omega$. This, clearly, is an E-parameter associated with the *ordered* Abelian semigroup

$$\mathbb{Z}_+^\omega := \{(k_i)_{i\in\omega} : k_i\in\mathbb{Z}_+\}; \tag{3.2}$$

recall that $\omega\in\kappa$ is a subset of $\mathbb{N}(d)$ inheriting its order. Then we let

$$\bar{\Phi}_\omega := ((\Phi_0)_\omega,\dots,(\Phi_n)_\omega). \tag{3.3}$$

The conditions of the theorem also involve operators S_ω defined on finitely supported functions $f:\mathbb{Z}_+^\omega\to\mathbb{R}$ by

$$(S_\omega f)(k) := \sum_{\ell\geqslant k} f(\ell), \quad k\in\mathbb{Z}_+^\omega. \tag{3.4}$$

We say that S_ω *is bounded in* $\bar{\Phi}$, if it can be continuously extended to $\Sigma(\bar{\Phi}_\omega)$ and this extension belongs to $L(\bar{\Phi}_\omega)$. This, in particular, implies that for $f\in\Sigma(\bar{\Phi}_\omega)$

$$\sum_\ell |f(\ell)| < +\infty. \tag{3.5}$$

Given $(\bar{X}, \mathcal{A}^i)$, $1 \leqslant i \leqslant d$, $\bar{\Phi}$ and κ, one introduces now the *assumptions* for the main theorem.

(a) Each $(\bar{X}, \mathcal{A}^i)$ is *complemented* and $\mathcal{P}^i := \{P_k^i \in L(\bar{X}) : k \in \mathbb{Z}_+\}$ is the corresponding family of projections.
(b) Projections of distinct families $\mathcal{P}^i$ commute.
(c) Each operator S_ω with $\omega \in \kappa$ is bounded in $\bar{\Phi}$.

Under these conditions the following is true.

Theorem 3.1. *If an interpolation functor $\mathcal{F}$ splits each n-tuple $\bar{\Phi}_\omega(\bar{X})$, $\omega \in \kappa$, then it also splits the n-tuple $E_{\bar{\Phi}}(\bar{X}; \mathcal{A})$ with $\mathcal{A}$ defined by* (3.1), *that is to say,*

$$\mathcal{F}(E_{\bar{\Phi}}(\bar{X}; \mathcal{A})) = E_{\mathcal{F}(\bar{\Phi})}(\mathcal{F}(\bar{X}); \mathcal{A}) \tag{3.6}$$

with equivalence of the norms.

Proof. We begin with the following auxiliary result. Let $\mathcal{A}^\omega := \{A_k : k \in \mathbb{Z}_+^\omega\}, \omega \in \kappa$, be AF in $\bar{X}$ defined by

$$A_k := \sum_{i \in \omega} A_{k_i}^i. \tag{3.7}$$

In this case $e_{\mathcal{A}^\omega}$, see (2.3), is a function of variables k_i, $i \in \omega$. Let us consider an n-tuple of AS's $E_{\bar{\Phi}_\omega}(\bar{X}; \mathcal{A}^\omega)$; we simplify this notation by putting

$$E_{\bar{\Phi}}^\omega(\bar{X}; \mathcal{A}) := E_{\bar{\Phi}_\omega}(\bar{X}; \mathcal{A}^\omega). \tag{3.8}$$

Proposition 3.2. *There exists a morphism R_ω in the category $\mathcal{B}_n$ that maps $E_{\bar{\Phi}}^\omega(\bar{X}; \mathcal{A})$* in *$\bar{\Phi}_\omega(\bar{X})$ and possesses a right inverse morphism.*

Proof. Set for $k \in \mathbb{Z}_+^\omega$

$$P_k := 1 - \prod_{i \in \omega} (1 - P_{k_i}^i). \tag{3.9}$$

The commutativity condition (b) implies that $P_k a = a$ for $a \in A_{k_i}^i$, $i \in \omega$. Hence P_k is a projection on $\sum_{i \in \omega} A_{k_i}^i =: A_k$, $k \in \mathbb{Z}_+^\omega$, and the following is true.

Lemma 3.3. *$(\bar{X}, \mathcal{A}^\omega)$ is complemented and $\mathcal{P}^\omega := \{P_k : k \in \mathbb{Z}_+^\omega\}$ is the corresponding family of projections.*

To introduce the required morphism R_ω one sets for $k \in \mathbb{Z}_+^\omega$

$$R_k := \prod_{i \in \omega} P_{k_i}^i \quad \text{and} \quad R_k^j := \prod_{i \in \omega \backslash \{j\}} P_{k_i}^i. \tag{3.10}$$

By the commutativity condition

$$R_k = P_{k_i}^i R_k^i, \quad i \in \omega. \tag{3.11}$$

Since P_0^i is a projection on $A_0^i := \{0\}$, see (2.1), one also gets

$$R_k = 0, \quad \text{if} \quad \min_{i\in\omega} k_i = 0. \tag{3.12}$$

Define now R_ω on finitely supported vector-functions $f : \mathbb{Z}_+^\omega \to X$ by

$$R_\omega f := \sum_{k\in\mathbb{Z}_+^\omega} R_{k+1} f(k); \tag{3.13}$$

here $k+1 := (k_i+1)_{i\in\omega}$ and X is an intermediate space of $\bar{X}$.

Let now $\mathcal{G}$ be an interpolation functor on $\mathcal{B}_n$; let also

$$\Phi := \mathcal{G}(\bar{\Phi}_\omega) \quad \text{and} \quad X := \mathcal{G}(\bar{X}). \tag{3.14}$$

We will show that R_ω is defined by (3.13) for all $f\in\Phi(X)$. In fact, by the condition (c), $S_\omega \in L(\bar{\Phi}_\omega)$ and therefore the interpolation inequality (2.11) yields

$$||S_\omega||_\Phi \leqslant C_\mathcal{G} ||S_\omega||_{\bar{\Phi}_\omega} < +\infty. \tag{3.15}$$

This, in turn, implies for $f\in\Phi(X)$ the inequality

$$\sum_{k\in\mathbb{Z}_+^\omega} ||f(k)||_X < +\infty,$$

see (3.5). At last, $R_k \in L(\bar{X})$ and therefore

$$||R_k||_X \leqslant C_\mathcal{G} ||R_k||_{\bar{X}} \leqslant C_\mathcal{G} \prod_{i\in\omega} ||\mathcal{P}^i||_{\bar{X}}. \tag{3.16}$$

Hence the series in (3.13) is absolutely convergent and R_ω is defined for all $f\in\Phi(X)$.

Lemma 3.4. *R_ω maps $\Phi(X)$ in $E_\Phi(X;\mathcal{A}^\omega)$ and its norm is bounded by a constant dependent only on the amounts in the right-hand side of* (3.15) *and* (3.16).

Proof. For $k\in\mathbb{Z}_+^\omega$ one sets

$$[k] := \mathbb{Z}_+^\omega \setminus \{\ell\in\mathbb{Z}_+^\omega : \ell_i > k_i - 1, \quad i\in\omega\}.$$

Let us show that the element

$$y_k := \sum_{\ell\in[k]} R_{\ell+1} f(\ell), \quad f\in\Phi(X)$$

belongs to A_k. To this end one presents $[k]$ as the disjoint union of the sets $\Omega_0 := \{\ell\in\mathbb{Z}_+^\omega : \ell_i < k_i\}$ and $\Omega_i := \{\ell\in\mathbb{Z}_+^\omega : \ell_i \geqslant k_i - 1 \text{ and } \ell_j < k_j,\ j\neq i\}$, $i\in\omega$. By (3.11),

$$\sum_{\ell\in\Omega_0} R_{\ell+1} f(\ell) = \sum_{i\in\omega} \sum_{\ell_i<k_i} P_{\ell_i+1}^i(z_\ell^i),$$

where z_ℓ^i are suitable elements from X. The latter sum, clearly, belongs to $\sum_{i\in\omega}\sum_{\ell_i\leqslant k_i} A_{\ell_i}^i \subset \sum_{i\in\omega} A_{k_i}^i =: A_k$, see (3.7) and (2.1). In turn, for a suitable $z_i\in X$

$$\sum_{i\in\omega}\sum_{\ell\in\Omega_i} R_{\ell+1} f(\ell) = \sum_{i\in\omega} P_{k_i}^i(z_i) \in A_k.$$

Hence $y_k \in A_k$ and therefore

$$e_k(R_\omega f; X) = e_k\left(\sum_{i\in\omega}\sum_{\ell_i > k_i - 1} R_{\ell+1} f(\ell); X\right)$$
$$\leqslant \sup_\ell ||R_\ell||_X \sum_{\ell \geqslant k} ||f(\ell)||_X =: CS_\omega(||f||_X)(k).$$

Here and below C stands for a constant depending only on unessential parameters that can vary from line to line. Taking here Φ-norm and using the inequalities (3.16) and (3.15), one gets

$$||R_\omega f||_{E_\Phi(X;\mathcal{A}^\omega)} := ||(e_k(Rf;X))_{k\in\mathbb{Z}_+^\omega}||_\Phi$$
$$\leqslant c||S_\omega||_\Phi\, ||f||_{\Phi(X)} \leqslant c||f||_{\Phi(X)}.$$

The proof is complete. $\square$

Apply this lemma to the case of $\mathcal{G}$ equal to $\Sigma : \bar{X} \to \Sigma(\bar{X})$ and $\pi_i : \bar{X} \to X_i,\ 0 \leqslant i \leqslant n$, respectively. Then it implies that R_ω is a morphism of the category $\mathcal{B}_n$ mapping $E^\omega_{\bar{\Phi}}(\bar{X};\mathcal{A})$ in $\bar{\Phi}_\omega(\bar{X})$. Besides, its norm is bounded by a constant depending only on $||S_\omega||_{\bar{\Phi}}$ and $||\mathcal{P}^i||_{\bar{X}},\ i \in \omega$.

To complete the proof of Proposition 3.2, it remains to introduce a morphism which is right inverse to R_ω. To this end one defines the operator of mixed differences Δ given on the family of projectors $P_k \in \mathcal{P}^\omega$, see (3.9),

$$\Delta P_k := \prod_{i\in\omega} (P_{k+e_i} - P_k), \tag{3.17}$$

where $\{e_i\}_{i=1}^d$ is the standard basis of $\mathbb{R}^d$. By commutativity of P_k's this can be rewritten in the form

$$\Delta P_k = \sum_{v\in\{0,1\}^\omega} (-1)^{\varepsilon(v)} P_{k+v}, \tag{3.18}$$

where $\varepsilon(v)$ is the number of zero coordinates of v.

Define now, up to sign, the required right inverse P_ω on elements of $\Sigma(E^\omega_{\bar{\Phi}}(\bar{X};\mathcal{A}))$ by

$$P_\omega x := (\Delta P_k)_{k\in\mathbb{Z}_+^\omega}. \tag{3.19}$$

Under the notations of Lemma 3.4 the following is true.

Lemma 3.5. *P_ω maps $E_\Phi(X;\mathcal{A}^\omega)$ in $\Phi(X)$ and its norm is bounded by a constant depending only on $C_{\mathcal{G}}$, $||S_\omega||_{\bar{\Phi}}$ and $||\mathcal{P}^i||_{\bar{X}},\ i \in \omega$.*

Proof. According to (2.7)

$$||x - P_k x||_X \leqslant (1 + ||P_k||_X) e_k(x; X). \tag{3.20}$$

Together with (3.18) this gives

$$||\Delta P_k||_X \leqslant \left(1 + \sup_k ||P_k||_X\right) \sum_{v \in \{0,1\}^\omega} e_{k+v}(x; X).$$

Since the translation operators $f(k) \to f(k+v),\ v \in \{0,1\}^\omega$ are bounded on the set of positive f by $S_\omega f$, their norms in Φ are at most $||S_\omega||_\Phi$. Therefore

$$\begin{aligned} ||P_\omega x||_{\Phi(X)} &:= ||||(\Delta P_k x)_{k \in \mathbb{Z}_+^\omega}||_X||_\Phi \\ &\leqslant 2^{|\omega|}\left(1 + \sup_k ||P_k||_X\right) ||S_\omega||_\Phi ||(e_k(x; X))_{k \in \mathbb{Z}_d^\omega}||_\Phi \\ &=: C||x||_{E_\Phi(X; \mathcal{A}^\omega)}. \end{aligned}$$

By (3.15) and (3.16) C is bounded by a constant depending only on the desired parameters. $\square$

Using, as before, this lemma for the case of $\mathcal{G}$ equal to Σ and $\pi_i,\ i \in \omega$, one concludes that P_ω is a morphism of $\mathcal{B}_n$ which maps $E^\omega_{\bar{\Phi}}(\bar{X}; \mathcal{A})$ in $\bar{\Phi}_\omega(\bar{X})$, and its norm is bounded by a constant depending only on $||S_\omega||_{\bar{\Phi}}$ and $||\mathcal{P}^i||_{\bar{X}},\ i \in \omega$.

Let us now establish that, up to sign, the morphism P_ω is a right inverse to R_ω. It suffices to prove that

$$R_\omega P_\omega x = (-1)^{d-1} x, \tag{3.21}$$

provided $x \in \Sigma(E^\omega_{\bar{\Phi}}(\bar{X}; \mathcal{A}))$. To this end one begins with the identity

$$\sum_{\ell \leqslant k} \Delta R_\ell x = \sum_{v \in V_k} (-1)^{\varepsilon(v)} R_v x, \quad k \in \mathbb{Z}_+^\omega, \tag{3.22}$$

where V_k is the set of vertices of the parallelepiped $\{x \in \mathbb{R}^\omega : 0 \leqslant x_i \leqslant k_i\}$ and $\varepsilon(v)$ is the number of zero coordinates of v. Since $R_v = 0$ if $\min_{i \in \omega} v_i = 0$ (see (3.12)), the right-hand side of (3.22) equals $R_k x = (R_k x - x) + x$. Besides, by (3.10) and (3.20), we obtain for $X := \Sigma(E^\omega_{\bar{\Phi}}(\bar{X}; \mathcal{A}))$ the inequality

$$||R_k x - x||_X \leqslant \sum_{i \in \omega} \left(\prod_{j > i} ||P^j_{k_j}||_X\right) \cdot ||x - P^i_{k_i} x||_X \leqslant C \sum_{i \in \omega} e_{k_i}(x; X),$$

with C depending only on $||\mathcal{P}^i||_{\bar{X}},\ i \in \omega$, see (3.16) for $\mathcal{G} := \Sigma$. Since $A^i_{k_i} = A_{k_i e_i}$ (see (3.7)) and $||S_\omega||_{\Sigma(\bar{\Phi}_\omega)} \leqslant ||S_\omega||_{\bar{\Phi}_\omega} < \infty$, the right-hand side of the above inequality tends to zero as each k_i becomes $+\infty$, see (3.5) for $f(\ell) := e_\ell(x; X)$ and $\Phi := \Sigma(\bar{\Phi}_\omega)$. Together with (3.22) this yields

$$\lim \sum_{\ell \leqslant k} \Delta R_\ell\, x = x \tag{3.23}$$

as $\min_{i \in \omega} k_i$ becomes infinity.

Now definitions (3.13) and (3.19) of R_ω and P_ω and the identity

$$\Delta P_k := \Delta\left(1 - \prod_{i\in\omega}(1 - P^i_{k_i})\right) = (-1)^{d-1}\Delta\left(\prod_{i\in\omega} P^i_{k_i}\right) := (-1)^{d-1}\Delta R_k$$

yield $R_\omega P_\omega = (-1)^{d-1}\,\Sigma\, R_{k+1}\Delta R_k$. Besides, (3.18) and the monotonicity of the family of projections R_k give

$$R_{k+1}\Delta R_k = \sum_{v\in\{0,1\}^\omega} (-1)^{\varepsilon(v)} R_{k+1}R_{k+v} = \sum_{v\in\{0,1\}^\omega} (-1)^{\varepsilon(v)} R_{k+v} = \Delta R_k.$$

Combining this with the previous identity and then applying (3.23), one gets

$$R_\omega P_\omega x = (-1)^{d-1}\sum_k \Delta R_k x = (-1)^{d-1}x.$$

Thus $(-1)^{d-1}P_\omega$ is a right inverse to R_ω.

Proposition 3.2 has been proved. $\square$

We continue the derivation of Theorem 3.1 by the following result. In its formulation $k_\omega := (k_i)_{i\in\omega}\in\mathbb{Z}^\omega_+$ whenever $k\in\mathbb{Z}^d_+$.

Lemma 3.6. *The equivalence*

$$e_k(x, X) \approx \sum_{\omega\in k} e_{k_\omega}(x; X) \tag{3.24}$$

holds with positive constants independent of $k\in\mathbb{Z}^d_+$ and $x\in X(:= \mathcal{G}(\bar{X}))$.

Proof. According to (3.1) and (3.7)

$$A_{k_\omega} \supset A_k := \bigcap_{\omega\in k}\left(\sum_{i\in\omega} A^i_{k_i}\right) := \bigcap_{\omega\in k} A_{k_\omega};$$

this implies the first inequality (3.24) as one gets

$$\sum_{\omega\in k} e_{k_\omega}(x; X) \leqslant |k| e_k(x; X).$$

To establish the inverse inequality, one introduces for a given $k\in\mathbb{Z}^d_+$ the operator

$$P_k := \prod_{\omega\in\kappa} P_{k_\omega}, \tag{3.25}$$

where P_{k_ω} is the projection on A_{k_ω} defined by (3.9). Since the operators of the product in (3.25) commute, P_k is a projection on $\bigcap_{\omega\in\kappa} A_{k_\omega} =: A_k$. Besides, $P_k\in L(\bar{X})$ and its norm is bounded by a constant depending only on $||\mathcal{P}^i||_{\bar{X}}$, $1\leqslant i\leqslant d$, and $P_k|_{X_i}$ is a projection on $A_k\cap X_i$, $0\leqslant i\leqslant n$. Therefore, $P_k|_X := P_k|_{\mathcal{G}(\bar{X})}$ belongs to $L(X)$, its norm is bounded by $||P_k||_{\bar{X}}$, and it is a projection on $A_k\cap X$. Thus, for $x\in X$

we have

$$e_k(x;X) \leqslant \|x - P_k x\|_X \leqslant C \sum_{\omega\in\kappa} \|x - P_{k_\omega} x\|_X$$
$$\leqslant C \sum_{\omega\in\kappa} (1 + \|P_{k_\omega}\|_X) e_{k_\omega}(x;X) \leqslant C \sum_{\omega\in\kappa} e_{k_\omega}(x;X),$$

where C depends only on $C_{\mathcal{G}}$ and $\|\mathcal{P}^i\|_{\bar{X}}$, $1 \leqslant i \leqslant d$. $\square$

We will now define the direct sum of the spaces $E^{\omega}_{\Phi}(X;\mathcal{A})$, $\omega\in\kappa$. Let us recall that the norm of $x := (x_\omega)_{\omega\in\kappa}$ in $\oplus_{\omega\in\kappa} E^{\omega}_{\Phi}(X;\mathcal{A})$ is given by

$$\|x\| := \sum_{\omega\in\kappa} \|x_\omega\|_{E^{\omega}_{\Phi}(X;\mathcal{A})}, \tag{3.26}$$

while $E^{\omega}_{\Phi}(X;\mathcal{A})$ is defined by (3.8) with $n = 0$. In this definition κ is assumed to be ordered in some way, and $X := \mathcal{G}(\bar{X})$.

Denote then by $D_\Phi(X)$ a "diagonal" of that direct sum comprising elements $(x_\omega)_{\omega\in\kappa}$ such that $x_\omega = x$ for all $\omega\in\kappa$ and some x from $\bigcap_{\omega\in\kappa} E^{\omega}_{\Phi}(X;\mathcal{A})$. It is easily seen that $D_\Phi(X)$ is a closed subspace of the direct sum.

Lemma 3.7. *The space $E_\Phi(X;\mathcal{A})$ is isomorphic to $D_\Phi(X)$ and the constants of isomorphism depend only on $C_{\mathcal{G}}$ and $\|\mathcal{P}^i\|_{\bar{X}}$, $1 \leqslant i \leqslant d$.*

Proof. In virtue of (3.24) and (3.8)

$$\|x\|_{E_\Phi(X;\mathcal{A})} := \|e_{\mathcal{A}}(x;X)\|_\Phi \approx \sum_{\omega\in\kappa} \|e_{\mathcal{A}^\omega}(x;X)\|_{\Phi_\omega} := \sum_{\omega\in k} \|x\|_{E^{\omega}_{\Phi}(X;\mathcal{A})}.$$

Recall that Φ_ω is the closed subspace of Φ comprising functions of variables k_i, $i\in\omega$. Then the mapping $I : x \mapsto (x_\omega)_{\omega\in\kappa}$, where $x_\omega := x$, yields the required isomorphism. $\square$

Using now the operators R_ω and P_ω of Proposition 3.2 one constructs the operators

$$R := \bigoplus_{\omega\in\kappa} R_\omega \quad \text{and} \quad P := \bigoplus_{\omega\in\kappa} P_\omega, \tag{3.27}$$

that is to say, P sends an element $(x_\omega)_{\omega\in\kappa}$ from $\oplus_{\omega\in\kappa} E^{\omega}_{\Phi}(X)$ in the element $(P_\omega x_\omega)_{\omega\in\kappa}$ while R acts similarly in the opposite direction. Using now the isomorphism $I : E_\Phi(X;\mathcal{A}) \to D_\Phi(X)$ from the previous lemma, one sets

$$\tilde{P} := PI, \quad \tilde{R} := I^{-1}R.$$

Then $\tilde{R} : \oplus_{\omega\in\kappa} \Phi_\omega \to E_\Phi(X;\mathcal{A})$ and $\tilde{P} : E_\Phi(X;\mathcal{A}) \to \oplus_{\omega\in\kappa} \Phi_\omega$, and the norms of these operators are bounded as required. Besides, by Proposition 3.2

$$\tilde{R}\tilde{P} = I^{-1}RPI = \pm 1_{E_\Phi(X;\mathcal{A})}. \tag{3.28}$$

Taking now $\mathcal{G}$ to be the functors $\bar{X} \to \Sigma(\bar{X})$ and $\bar{X} \to X_i$, $0 \leqslant i \leqslant n$, respectively, one establishes that the mapping $\tilde{R}$ is a morphism from $\oplus_{\omega\in\kappa} \bar{\Phi}_\omega(\bar{X})$ into $E_{\bar{\Phi}}(\bar{X};\mathcal{A})$, and $\tilde{P}$ is a morphism acting in the reverse direction. Recall that the n-tuple $\bar{\Phi}_\omega$ is defined by (3.3). Besides, by (3.28),

$$\tilde{R}\tilde{P} = \pm 1_{E_{\bar{\Phi}}(\bar{X};\mathcal{A})}.$$

We are now in a position to finalize the proof. Let $\mathcal{F}$ be an interpolation functor on $\mathcal{B}_n$ subject to the splitting condition

$$\mathcal{F}(\bar{\Phi}_\omega(\bar{X})) = \mathcal{F}(\bar{\Phi}_\omega)(\mathcal{F}(\bar{X})), \quad \omega \in \kappa. \tag{3.29}$$

Let $x \in \mathcal{F}(E_{\bar{\Phi}}(\bar{X};\mathcal{A}))$, and let $\Phi := \mathcal{F}(\bar{\Phi})$, $\Phi_\omega := \mathcal{F}(\bar{\Phi}_\omega)$ and $X := \mathcal{F}(\bar{X})$. By Proposition 2.7, the interpolation inequality (2.11) and the splitting condition (3.29) one has

$$\begin{aligned} ||\tilde{P}x||_{\mathcal{F}(\oplus_{\omega\in\kappa}\bar{\Phi}_\omega(\bar{X}))} &\leqslant C||\tilde{P}x||_{\bigoplus_{\omega\in\kappa}\mathcal{F}(\bar{\Phi}_\omega(\bar{X}))} \\ &\leqslant C||\tilde{P}x||_{\bigoplus_{\omega\in\kappa}\Phi_\omega(X)} \leqslant C||x||_{E_\Phi(X;\mathcal{A})}. \end{aligned}$$

Here C depends only on $C_{\mathcal{F}}$ and the norm of morphism $\tilde{P}$. Applying now (3.28) one obtains

$$||x||_{\mathcal{F}(E_{\bar{\Phi}}(\bar{X}\mathcal{A}))} = ||\tilde{R}\tilde{P}x||_{\mathcal{F}(E_{\bar{\Phi}}(\bar{X};\mathcal{A}))} \leqslant C||\tilde{P}x||_{\mathcal{F}(\oplus_{\omega\in\kappa}\bar{\Phi}_\omega(\bar{X}))}.$$

Together with the previous inequality this yields the embedding

$$E_\Phi(X;\mathcal{A}) := E_{\mathcal{F}(\bar{\Phi})}(\mathcal{F}(\bar{X});\mathcal{A}) \subset \mathcal{F}(E_{\bar{\Phi}}(\bar{X};\mathcal{A})).$$

The inverse embedding is derived in exactly the same fashion. Actually, one has

$$\begin{aligned} ||x||_{E_\Phi(X;\mathcal{A})} = ||\tilde{R}\tilde{P}x||_{E_\Phi(X;\mathcal{A})} &\leqslant C||\tilde{P}x||_{\bigoplus_{\omega\in\kappa}\Phi_\omega(X)} \\ &\leqslant C||\tilde{P}x||_{\mathcal{F}(\oplus_{\omega\in\kappa}\bar{\Phi}_\omega(\bar{X}))} \leqslant C||x||_{\mathcal{F}(E_{\bar{\Phi}}(\bar{X};\mathcal{A}))}, \end{aligned}$$

and the inverse embedding is also established.

The proof Theorem 3.1 is complete. $\square$

Concluding Remarks 3.8. (a) Theorem 3.1 remains to be true for quasicomplemented families $(\bar{X}, \mathcal{A}^i)$, $1 \leqslant i \leqslant d$, as well. The only change of the proof is as follows. We now define R_ω (see (3.13) by

$$R_\omega f := \sum_{k\in\mathbb{Z}_+^\omega} R_{k+2} f(k),$$

where $k+2 := (k_1+2, \dots, k_d+2)$. Then the set $[k]$ in the proof of Lemma 3.4 has to be defined as the complement of the set $\{\ell \in \mathbb{Z}_+^\omega : \ell_i > k_i - 2\}$.

Under this modification the derivation of Theorem 3.1 yields the required result for the quasicomplemented AF's.

(b) Strictly speaking, our definition of AS imposes one more assumption on the functor $\mathcal{F}$: the Banach lattice $\mathcal{F}(\bar{\Phi})$ is an E-parameter, provided that each Φ_i is. The functors used in the present paper do have this property. But we can remove this assumption in another way by modifying the definition of AS. In this new definition Φ is a Banach lattice satisfying the only condition (2.8), and $E_\Phi(X;\mathcal{A})$ is introduced by (2.10). Then $\mathcal{F}(\bar{\Phi})$ is a Banach lattice satisfying (2.8) but $E_\Phi(X;\mathcal{A})$ may be incomplete. We will say that $E_\Phi(X;\mathcal{A})$ is AS, if it is complete. Under this definition Theorem 3.1 holds without additional assumptions on $\mathcal{F}$.

4. Examples and applications

All our examples concern Besov spaces and their generalizations of various types. Their presentation as AS's is based on results going back to the classical S. Bernstein's and D. Jackson's theorems of trigonometric approximation. In what follows most of these results will be used in a form related to approximation in translation invariant Banach lattices rather than in L_p. Such a generalization does not require new ideas and can be derived by the very same argument as in the case of L_p, see, e.g., [Sh, Section 9.3], where a similar extension is introduced. It is worth pointing out that the interpolation theorems presented below remain to be new even in the case of L_p-spaces.

4.1. Isotropic Besov spaces

Let $\mathcal{B}$ be a one-parametric approximation family consisting of the linear subsets $A_0 := \{0\}$ and $A_k := B_{2^k}(\mathbb{R}^d),\ k = 0, 1, 2, \ldots$. Here $B_\sigma(\mathbb{R}^d)$ is the Bernstein class of bounded on $\mathbb{R}^d$ entire functions of exponential type and degree $\leqslant\sigma$. Then for the classical Besov space $B_p^{sq}(\mathbb{R}^d)$ the following is true:

$$B_p^{sq}(\mathbb{R}^d) = E_{\ell_q^s(\mathbb{Z}_+)}(L_p;\mathcal{B}), \tag{4.1}$$

where the E-parameter is given by

$$\|f\|_{\ell_q^s(\mathbb{Z}_+)} := \left\{\sum_{k=0}^{\infty} |2^{sk} f(k)|^q\right\}^{\frac{1}{q}}. \tag{4.2}$$

Besides, $\mathcal{B}$ is a quasicomplemented in an n-tuple $(L_{p_0}, \ldots, L_{p_n})$ and the respective quasiprojectors P_k are the (Vallee–Poussen) operators, that is to say,

$$P_k f := \psi_{2^k} * f, \tag{4.3}$$

where $\psi_t(x) = t^{-d}\psi(tx),\ t>0$, and its Fourier transform $\hat{\psi}$ is a test function supported by the Euclidean ball $\{\xi\in(\mathbb{R}^d)^* : |\xi|\leqslant 2\}$ and equals 1 if $|\xi|\leqslant 1$. The proofs are presented, e.g., in [N]; they can be easily adapted to the case of translation invariant Banach lattices possessing the Fatou property. The only essential fact that should be used in this adaptation is the generalized Minkowski inequality for a Banach lattice X with the Fatou property. It asserts that

$$\left\| \int_{\mathbb{R}^d} f(x,y)\,d\mu(y) \right\|_X \leqslant \int_{\mathbb{R}^d} \|f(\cdot,y)\|_X\, d\mu(y),$$

for a nonnegative bounded Borel measure μ and the function $x \to f(x,y)$ belonging to X for μ-almost all y, see, e.g., [KPS, Section II.3].

So, in the sequel X and $\bar{X}$ are, respectively, a translation invariant Banach lattice on $\mathbb{R}^d$ with the Fatou property and an n-tuple of such lattices. It is more convenient for now to write $B^{sq}(L_p(\mathbb{R}^d))$ instead of $B_p^{sq}(\mathbb{R}^d)$ and so on. Hence we have , under this notation,

$$B^{sq}(X) = E_{\ell_q^s(\mathbb{Z}_+)}(X; \mathcal{B}). \tag{4.4}$$

Besides, $\mathcal{B}$ is quasicomplemented in $\bar{X}$ and the respective quasiprojectors are given by (4.3). These facts lead to the following results.

Corollary 4.1. *For the upper complex method* $\mathcal{C}^\theta$ *on* $\mathcal{B}_n$, *see* (2.23), *the isomorphism*

$$\mathcal{C}^\theta(B^{s_0q_0}(X_0), \ldots, B^{s_nq_n}(X_n)) = B^{sq}(X) \tag{4.5}$$

holds with

$$s := \sum_{i=0}^{n} \theta_i s_i, \quad q^{-1} := \sum_{i=0}^{n} \theta_i q_i^{-1} \quad \textit{and} \quad X := \bar{X}^\theta. \tag{4.6}$$

Proof. By (4.4), the left-hand side of (4.5) equals $C^\theta(E_{\bar{\Phi}}(\bar{X}; \mathcal{B}))$ with $\Phi_i := \ell_{q_i}^{s_i}(\mathbb{Z}_+)$, $1 \leqslant i \leqslant d$. Since the functor C^θ splits $\bar{\Phi}(\bar{X})$, see (2.26), one can apply the variant of Theorem 3.1 presented in Remark 3.8 (b). In this case $\bar{\mathcal{A}}$ consists of the single family $\mathcal{B}$, i.e., $d = 1$ and $k = \mathbb{N}(1) = \{1\}$. The corresponding integral operator $S_{\{1\}}$ is given by

$$(S_{\{1\}}f)(k) := \sum_{\ell=k}^{\infty} f(\ell), \ \ k \in \mathbb{Z}_+. \tag{4.7}$$

The Hölder inequality implies that $S_{\{1\}}$ is bounded in $\ell_q^s(\mathbb{Z}_+)$, if $s > 0$ and $1 \leqslant q \leqslant \infty$. Hence the aforementioned variant of Theorem 3.1 implies the equality

$$\mathcal{C}^\theta(E_{\bar{\Phi}}(\bar{X}; \mathcal{B})) = E_{\bar{\Phi}^\theta}(\mathcal{C}^\theta(\bar{X}); \mathcal{B}).$$

To complete the proof it remains to show that

$$\mathcal{C}^\theta(\bar{X}) = \bar{X}^\theta \quad \text{and} \quad \bar{\Phi}^\theta = \ell_q^s(\mathbb{Z}_+),$$

where s and q are given in (4.6). The former equality follows from the validity of this result for couples of Banach lattices with the Fatou property [C] and induction on n, cf. the proof of Proposition 2.11. The latter one follows, for $n = 1$, from (2.20) and the Hölder inequality; the case $n > 1$ is then derived by induction.

Combining now all these facts with (4.4), one gets

$$\mathcal{C}^\theta(B^{s_0q_0}(X_0), \ldots, B^{s_nq_n}(X_n)) = E_{\ell_q^s}(\bar{X}^\theta; \mathcal{B}) = B^{sq}(\bar{X}^\theta).$$

The result has been proved. $\square$

Specially, for $X_i \coloneqq L_{p_i}$,

$$\bar{X}^\theta = L_p, \quad \text{where} \quad p^{-1} := \sum_{i=0}^{n} \theta_i p_i^{-1}, \tag{4.8}$$

see (2.31), and therefore

$$\mathcal{C}^\theta(B^{s_0 q_0}_{p_0}, \dots, B^{s_n q_n}_{p_n}) = B^{sq}_p, \tag{4.9}$$

where θ-means s, p, q are given by (4.6) and (4.8).

Let us now apply the very same argument to the case of the real interpolation functor $\mathcal{R}_{\theta q}$, see (2.30), and use the result asserting that

$$\mathcal{R}_{\theta q}(L_{p_0}, \dots, L_{p_n}) = L_{pq}, \tag{4.10}$$

where p is as above and L_{pq} is a Lorentz space. Formula (4.10) can be straightforwardly derived from Proposition 9.3 and Theorem 9.3 from [Sp], see also [E], where this was established for $n = 2$ in another way. Then we immediately obtain

Corollary 4.2. *Under the above notations,*

$$\mathcal{R}_{\theta q}(B^{s_0 q_0}_{p_0}, \dots, B^{s_n q_n}_{p_n}) = B^{sq}(L_{pq}). \tag{4.11}$$

Specially, the right-hand side is B^s_q, if $q = p$.

4.2. Anisotropic Besov spaces

To avoid unessential but cumbersome details we confine ourselves to the case of periodic functions. So X is now a translation invariant Banach lattice with the Fatou property comprising measurable 2π-periodic in each variable functions on $\mathbb{R}^d$. The anisotropic Besov space over this X is determined by *smoothness* $\bar{s} \coloneqq (s_1, \dots, s_d)$ and the parameter $1 \leqslant q \leqslant \infty$ via the norm

$$||f||_{B^{\bar{s}q}(X)} \coloneqq ||f||_X + \sup_{1 \leqslant i \leqslant d} ||\omega^i_r(\cdot; f; X)||_{\ell^{s_i}_q(\mathbb{Z}_+)}; \tag{4.12}$$

here $r > \max\ s_i$ and

$$\omega^i_r(t; f; X) \coloneqq \sup_{0 < h \leqslant t} ||\Delta^r_{he_i} f||_X, \quad t > 0 \tag{4.13}$$

is the *partial r-modulus of continuity in direction* $e_i \coloneqq (\delta^i_j)^d_{j=1}$.

To present this as an AS, one introduces $\mathcal{T} \coloneqq \{T_k : k \in \mathbb{Z}^d_+\}$, where $T_0 \coloneqq \{0\}$ and T_k is the space of trigonometrical polynomials of degree at most 2^{k_i} in x_i, $1 \leqslant i \leqslant d$. The following presentation is a well-known classical result for $L_p(\mathbb{T}^d)$, see e.g., [T, Section 6.3.4], and can be easily extended to a general X. To formulate it we introduce E-parameter $\ell_\infty^{w_{\bar{s}}}$, see (2.29), where the weight is given by

$$w_{\bar{s}}(k) := \left(\sum_{i=1}^{d} 2^{-k_i s_i}\right)^{-1}, \quad k \in \mathbb{Z}_+^d. \tag{4.14}$$

Then it is true that

$$B^{\bar{s},\infty}(X) = E_{\ell_\infty^{w_{\bar{s}}}}(X; \mathcal{T}). \tag{4.15}$$

The family $\mathcal{T}$, in turn, is the intersection of families $\mathcal{T}^i := \{T_k^i : k \in \mathbb{Z}_+\}$, $1 \leqslant i \leqslant d$, that is to say,

$$T_k = \bigcap_{i=1}^{d} T_{k_i}^i, \quad k \in \mathbb{Z}_+^d. \tag{4.16}$$

Here $T_0^i := \{0\}$ and T_k^i is the space of *quasipolynomials with respect to* x_i of degree at most 2^k, $k \geqslant 1$. In other words, functions of this space are trigonometric polynomials in x_i of degree 2^k, the coefficients of which are functions from $L_1(\mathbb{T}^d)$ *independent* of x_i.

Furthermore, each $\mathcal{T}^i$ is quasicomplemented in an arbitrary n-tuple of Banach lattices of the type considered here. The required quasiprojectors P_k^i, $k \geqslant 1$, are introduced as follows. Let V_N be the classical Vallee–Poussen operator mapping 2π-periodic univariate functions into trigonometric polynomials of degree $2N$ and preserving polynomials of degree N, see e.g., [T]. Then, for $f \in L_1(\mathbb{T}^d)$ the quasipolynomial $P_k^i f \in T_{k+1}^i$ is the result of applying V_{2^k} to f regarded as a function x_i with the fixed remaining variables. It is well-known for $L_p(\mathbb{T}^d)$ and can be easily extended to the general case that

$$\sup\{\|P_k^i\|_X : k \in \mathbb{Z}_+\} < \infty.$$

Besides, by the definition, P_k^i and $P_{k'}^{i'}$ commute if $i \neq i'$.

Hence we are now in a position to apply the variant of Theorem 3.1 from Remark 3.8 (a). In this case κ is the partition of $\mathbb{N}(d)$ into one point subsets $\{i\}$, $1 \leqslant i \leqslant d$, and

$$\bar{\Phi} = (\ell_\infty^{w_{\bar{s}_0}}(\mathbb{Z}_+^d), \ldots, \ell_\infty^{w_{\bar{s}_n}}(\mathbb{Z}_+^d)),$$

see (4.15). Note that for $\Phi := \ell_\infty^{w_{\bar{s}}}$ the corresponding E-parameter $\Phi_{\{i\}}$ consists of functions from Φ depending only on k_i. Hence

$$\Phi_{\{i\}} = \ell_\infty^{s_i}(\mathbb{Z}_+),$$

see (4.2) and (4.14). The corresponding n-tuple $\bar{\Phi}_{\{i\}}$ is formed by E-parameters $\ell_\infty^{s_{ji}}(\mathbb{Z}_+^d)$, provided $\bar{s}_j := (s_{ji})_{i=1}^d$.

At last, the operator $S_{\{i\}}$ from (3.4) with $\omega := \{i\}$ coincides with that in (4.7). Therefore $S_{\{i\}}$ is bounded in each $\ell_q^s(\mathbb{Z}_+)$ with $s > 0$ and $1 \leqslant q \leqslant \infty$ and it then follows that $S_{\{i\}}$ is bounded in $\bar{\Phi}$.

Applying now the aforementioned variant of Theorem 3.1 to the case (4.15) and repeating with a trivial modification the argument of proof for Corollary 4.1, we immediately get the next relation:

$$C^{\theta}(B^{\bar{s}_0,\infty}(X_0),\ldots,B^{\bar{s}_n,\infty}(X_n)) = E_{\ell^w_\infty(\mathbb{Z}^d_+)}(\bar{X}^\theta;\mathcal{T}), \tag{4.17}$$

where

$$w := \prod_{i=0}^{n} (\omega_{\bar{s}_i})^{\theta_i}. \tag{4.18}$$

The similar argument can be applied in the case of the real interpolation method.

Remark 4.3. It is easy to show that the right-hand side of (4.17) embeds into $B^{\bar{s},\infty}(\bar{X}^\theta)$ with $\bar{s}$ given by (4.6).

Remark 4.4. To apply Theorem 3.1 to the general situation, we have to present the space $B^{\bar{s},q}(X)$ with arbitrary q as a d-parametric AS of a form $E_{\ell^\omega_q(\mathbb{Z}^d_+)}(X)$. Surprisingly, we have to solve a nonlinear algebraic equation to find the parameters for ω.

4.3. Spaces with dominated mixed difference

In this subsection we also consider only the case of periodic functions leaving to the reader the case of functions on $\mathbb{R}^d$. Besides, the basic approximation facts related to our interpolation result were proved in [B] only for $X := L_p(\mathbb{T}^d)$, $1 \leqslant p \leqslant \infty$, and their extension to more general Banach lattices requires some additional argument (see concluding remarks).

To introduce our main object, we need the notion of *mixed modulus of continuity of order* $k \in \mathbb{Z}^d_+$. Recall that it is given for $f \in L_p(\mathbb{T}^d)$ by

$$\omega_k(t;f;L_p) := \sup_{0 \leqslant h \leqslant t} \|\Delta^k_h f\|_p, \quad t \in \mathbb{R}^d_+. \tag{4.19}$$

The mixed difference of this definition is introduced as a product of partial differences, i.e.,

$$\Delta^k_h := \prod_{i=1}^{d} \Delta^{k_i}_{he_i}.$$

The space of interest $\Lambda^{\bar{s}q}_p$ with $\bar{s} = (s_1,\ldots,s_d)$, $s_i > 0$ and $1 \leqslant p$, $q \leqslant \infty$ consists of functions $f \in L_p(\mathbb{T}^d)$ whose norms

$$\|f\|_{\Lambda^{\bar{s}q}_p} := \|f\|_p + \|\omega_k(\cdot;f;L_p)\|_{\ell^{\bar{s}}_q(\mathbb{Z}^d_+)} \tag{4.20}$$

are finite. Here $k_i > s_i$, $1 \leqslant i \leqslant d$, and the E-parameter on the right is given through the norm

$$\|f\|_{\ell^{\bar{s}}_q(\mathbb{Z}^d_+)} := \left\{ \sum_{\ell \in \mathbb{Z}^d_+} \left| 2^{\ell\cdot\bar{s}} f(\ell)\right|^q \right\}^{\frac{1}{q}}.$$

The basic approximation result related to this space involves the d-parametric AF $\mathcal{TQ} := \{TQ_k : k \in \mathbb{Z}_+^d\}$ with

$$TQ_k := \sum_{i=1}^{d} T_i^{k_i},$$

where $T_k^i,\ k \in \mathbb{Z}_+$ is the space of quasipolynomials introduced above, see (4.16) and the subsequent text. The required result asserts that up to equivalence of the norms

$$\Lambda_p^{\bar{s}q} = E_{\ell_q^{\bar{s}}(\mathbb{Z}_+^d)}(L_p; \mathcal{TQ}), \tag{4.21}$$

see [B].

In turn, $\mathcal{TQ}$ is quasicomplemented in each n-tuple $(L_{p_0}, \dots, L_{p_n})(\mathbb{T}^d)$, $1 \leqslant p_i \leqslant \infty$, and the corresponding family of quasiprojectors is introduced by

$$P_k := 1 - \prod_{i=1}^{d} (1 - P_{k_i}^i), \quad k \in \mathbb{Z}_+^d,$$

where $P_k^i,\ k \in \mathbb{Z}_+$ were defined in the previous subsection via the Vallee–Poussen operators.

Hence we can now apply, as before, the variant of Theorem 3.1. In this case the covering κ consists of the single set $\mathbb{N}(d)$, and the corresponding operator

$$(S_{\mathbb{N}(d)}f)(k) := \sum_{\ell \geqslant k} f(\ell), \quad k \in \mathbb{Z}_+^d$$

is bounded in each space $\ell_q^{\bar{s}}(\mathbb{Z}_+^d)$ with $s_i > 0$ and $1 \leqslant q \leqslant \infty$, by the Hölder inequality. Using the same argument as in the previous cases, we then immediately obtain the following results.

Corollary 4.5. *Up to equivalence of the norms*

$$\mathcal{C}^\theta(\Lambda_{p_0}^{\bar{s}^0 q_0}, \dots, \Lambda_{p_n}^{\bar{s}^n q_n}) = \Lambda^{\bar{s}q}, \tag{4.22}$$

where θ-means $\bar{s}, p, q$ are introduced as in (4.6) *and* (4.8).

Corollary 4.6. *Up to equivalence of the norms*

$$\mathcal{R}_{\theta q}(\Lambda_{p_0}^{\bar{s}^0 q_0}, \dots, \Lambda_{p_n}^{\bar{s}_n q_n}) = \Lambda_{pq}^{\bar{s}q}, \tag{4.23}$$

where $\bar{s}, p, q$ are as above, and the space on the right-hand side is defined by (4.19) *and* (4.20) *with L_p replaced by the Lorentz space L_{pq}.*

Particularly, the right-hand side is $\Lambda_p^{\bar{s}q}$, if $q = p$.

4.4. Concluding remarks

(a) Using approximation by univariate splines with uniformly distributed knots and their tensor products, one can extend all the interpolation results presented above to the case of nonperiodic functions defined on d-cubes, bounded or unbounded.

(b) Derivation of (4.21) presented in [B] can be easily adapted to the case of mixed $L_{\bar{p}}$ space with $\bar{p} := (p_1, \ldots, p_d)$. Hence (4.22) remains valid with p_i replaced by vectors $\bar{p}^i$. The resulting space $\Lambda_{\bar{p}}^{\bar{s}\bar{q}}$ with $\bar{p}$ being θ-mean of $\bar{p}^i$ is defined by (4.20) with L_p replaced by $L_{\bar{p}}$.

(c) Using the Calderón–Mityagin interpolation theorem [M], it is possible to extend (4.21) to the case of rearrangement invariant spaces over $\mathbb{T}^d$ or d-cube and in this way to extend Corollary 4.5 to this class of Banach lattices.

(d) At last, one can consider in the same fashion the case of Λ-spaces determined by a given set of mixed moduli of continuity.

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